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The field boundary of two line currents immersed in a streaming plasma

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The cavity in which the magnetic field of two arbitrary line currents is confined by a streaming plasma which is assumed cold and perfectly conducting is investigated by using conformal transformations. When the magnetic field at the boundary is always directed in the same sense the finite breadth of the cavity at infinity depends only on the algebraic sum of the inducing currents and not their position. If the two line currents are of opposite sign the boundary magnetic field may change sign at two 'pseudo-singularities'.

1. Introduction

The cavity in which sources of magnetic field are confined by stationary or streaming plasmas has been investigated by several authors in an attempt to estimate theoretically the shape and size of the cavity in which the earth's magnetic field is confined by the solar plasma. The problem of determining the cavity in which a three-dimensional dipole is confined by a stationary or streaming plasma is very difficult and has been investigated only by approximate or numerical methods (see, for example, Beard 1960; Midgley & Davis 1962; Slutz 1962; Midgley & Davis 1963; Mead & Beard 1964). Fortunately we can obtain exact solutions for two-dimensional models by using conformal transformations.

Hurley (1961*a*) solved the problem of a plasma streaming past the magnetic field of a line current. Dungey (1961), Hurley (1961*b*) and Zhigulev & Romishevskii (1959) independently solved the problem of a plasma streaming past a two-dimensional dipole. Sozou (1964) investigated the problem of a two-dimensional dipole and also that of two arbitrary line currents (1966) immersed in a plasma at uniform pressure.

In this note we investigate the problem of two arbitrary line currents immersed in a streaming plasma. Thus the problems considered by Hurley, Dungey and Zhigulev & Romishevskii are special cases of the present problem. It is assumed that the distance between the line currents and the direction of incidence of the streaming plasma are such that both currents lie in one cavity at the boundary of which there is a thin current sheath. Thus plasma and magnetic field occupy different positions in space and are shielded from each other by the current sheath.

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Using conformal transformations we obtain an exact solution. Given the position of the line currents in the physical plane in the general case, we obtain a number of complicated equations (the number of equations varies between one and ten depending on the sign and position of the line currents) in an equal number of parameters whose determination enables us to calculate the boundary. Thus it is simpler to prescribe the position of the line currents in the transformed plane and proceed to obtain the corresponding boundary and position of the line currents in the physical plane.

2. Equations of the problem

Let the magnetic field be $\mathbf{B} = (B_x, B_y, 0)$. This must satisfy the following conditions

$$\left.\begin{array}{ll} \nabla . \mathbf{B} = 0 & \text{everywhere,} \\ \nabla \times \mathbf{B} = 0 & \text{everywhere inside the cavity except at the} \\ & \quad\text{line currents.} \end{array}\right\} \tag{1}$$

If v is the undisturbed speed of the stream, ρ its density and ϕ the angle between the normal to the boundary and the stream then an amount of momentum $\rho v \cos\phi$ times $v\cos\phi$ is incident normally to the boundary per unit time. Thus the particle pressure is $k\rho v^2 \cos^2\phi$, where $k = 2$ if we have specular reflexion and $k = 1$ if the stream particles are brought to rest at the boundary and thence proceed tangentially. The magnitude of k, which is assumed a fixed constant throughout, affects the linear dimensions but not the shape of the boundary. At the boundary the particle pressure is equal to the magnetic pressure, that is

$$|\mathbf{B}|^2 = B_x^2 + B_y^2 = p^2 \cos^2\phi, \tag{2}$$

where $p^2 = 8\pi k \rho v^2$.

Equation (1) is satisfied if $\bar{B} = B_x - iB_y = \bar{B}(z)$ where $z = x + iy$ and a bar denotes complex conjugate.

Since at the boundary the magnetic field is tangential we have

$$\bar{B}dz = B_x dx + B_y dy = \mathbf{B}.d\mathbf{s} = \pm|\mathbf{B}|ds = \pm p\cos\phi\, ds = \pm p\,dy, \tag{3}$$

using (2), where ds is an element of arc of the boundary. In (3) the positive sign holds when $\mathbf{B}$ and $d\mathbf{s}$ are in the same direction and the negative sign holds when $\mathbf{B}$ and $d\mathbf{s}$ are in opposite directions. Let

$$\bar{B} = d\Phi/dz, \tag{4}$$

and let the two line currents be I_1 and I_2 situated at a_1 and a_2 respectively. It is assumed that the distance between the line currents and the direction of the streaming plasma are such that the two line currents are confined in one cavity.

If we assume that very near the line currents the lines of force are not affected substantially by the incident stream pressure we require

$$\left.\begin{aligned} d\Phi/dz = \bar{B}(z) &\sim 2I_1/\{i(z-a_1)\}, \quad \text{when} \quad z \to a_1, \\ &\sim 2I_2/\{i(z-a_2)\}, \quad \text{when} \quad z \to a_2. \end{aligned}\right\} \tag{5}$$

Let us assume (Riemann's mapping theorem) that there is a transformation $w(z)$ [and its inverse $z(w)$] that transforms the unknown domain in the z-plane into a unit circle in the w-plane so that the origins in the two domains correspond. Let also the points a_1 and a_2 in the z-plane correspond to points A_1 and A_2 in the w-plane, that is

$$\left.\begin{aligned} w(a_1) = A_1 \quad \text{or} \quad z(A_1) = a_1, \\ w(a_2) = A_2 \quad \text{or} \quad z(A_2) = a_2. \end{aligned}\right\} \tag{6}$$

If we express Φ in terms of w we have

$$\Phi(z) = g(w). \tag{7}$$

From (5), (6) and (7) we get

$$\left.\begin{aligned} \frac{dg}{dw} = \frac{d\Phi}{dz}\frac{dz}{dw} \sim \frac{2I_1}{i(w-A_1)} \quad \text{as} \quad w \to A_1, \\ \sim \frac{2I_2}{i(w-A_2)} \quad \text{as} \quad w \to A_2, \end{aligned}\right\} \tag{8}$$

that is, the singularities of $d\Phi/dz$ in the z-plane are transformed into singularities of dg/dw at the corresponding points in the w-plane. Equations (3), (4) and (7) show that dg is real at the boundary. This and the fact that dg/dw has two simple poles at A_1 and A_2 inside the unit circle require that

$$\frac{dg}{dw} = -2i\left(\frac{I_1}{w-A_1} - \frac{\bar{A}_1 I_1}{\bar{A}_1 w - 1} + \frac{I_2}{w-A_2} - \frac{\bar{A}_2 I_2}{\bar{A}_2 w - 1}\right). \tag{9}$$

The mathematical formulation and equations of the problem up to this point are identical with the problem of two line currents immersed in a plasma at uniform pressure, which was considered by Sozou (1966), except for an extra factor of $\cos^2\phi$ in the boundary condition (2), which manifests itself in equation (3), and a slight change in notation (in order to avoid roots we denote the pressure by p^2 instead of p).

Equations (3), (4), (7) and (9) require that at the boundary

$$p\,dy = \pm 2\left[\frac{I_1(1-r_1^2)}{1+r_1^2-2r_1\cos(\theta_1-\theta)} + \frac{I_2(1-r_2^2)}{1+r_2^2-2r_2\cos(\theta_2-\theta)}\right]d\theta, \tag{10}$$

where $r_1 e^{i\theta_1} = A_1$ and $r_2 e^{i\theta_2} = A_2$. $dy = \mathrm{Im}([dz/dw]\,dw)$ and thus at the boundary ($dw = iw\,d\theta$)

$$dy = R\left(w\frac{dz}{dw}d\theta\right). \tag{11}$$

Since $w(z)$ transforms the domain in the z-plane conformally into a unit circle in the w-plane dw/dz (and dz/dw) is regular in the domain, that is, it has no zeros or singularities. Since the stream particles are assumed cold they cannot bend round to close the cavity, that is, the cavity extends to infinity. The breadth of the cavity at infinity is given by $y(\pi) - y(-\pi)$ which, as it is easily seen by integrating (10), is finite. Thus the boundary of the unit circle in the w-plane must have a singularity that corresponds to the point at infinity in the z-plane.

Thus we must find a function of w, (dz/dw), which is regular inside the unit circle and has one singularity on the boundary, which will be the point $w = -1$ if the plasma is coming from the positive direction of the x-axis, corresponding to the point at infinity in the z-plane. From (10) and (11) we find that at the boundary ($w = e^{i\theta}$),

$$R\left(w\frac{dz}{dw}\right) = \pm\frac{2}{p}\left[\frac{I_1(1-r_1^2)}{1+r_1^2-2r_1\cos(\theta_1-\theta)}+\frac{I_2(1-r_2^2)}{1+r_2^2-2r_2\cos(\theta_2-\theta)}\right]. \tag{12}$$

If $I_2 = 0$, all the above conditions are satisfied by

$$\frac{dz}{dw} = -\frac{4I_1(1+\bar{A}_1)}{p(w+1)(\bar{A}_1w-1)}. \tag{13}$$

From this equation we recover the results contained in Hurley's paper (1961*a*).

Depending on $r_1, \theta_1, I_1, r_2, \theta_2$, and I_2 (10) may have two or no zeros as θ varies between $-\pi$ and π. If (10) has no zeros, the surface field is always directed in the same sense. If (10) has two zeros, say $\theta = \alpha_1$ and $\theta = \alpha_2$, $(-\pi \leqslant \alpha_2 \leqslant \alpha_1 \leqslant \pi)$, the boundary is parallel to the direction of the stream at the corresponding points in the z-plane. These two points are connected by a line of force that passes between the line currents and divides the cavity into two regions, each of which is dominated by the magnetic field of one of the line currents. Thus the surface field changes direction at the corresponding points, say from anticlockwise between them to clockwise from them to infinity. In this case the plus sign in (12) holds between α_1 and α_2.

When the direction of the undisturbed stream is parallel to the line joining the line currents, A_1 and A_2 may be chosen to be real. In this case it can easily be seen from (9) or (10) that $\alpha_1 = -\alpha_2 = \alpha$ say. Equation (12) and the conditions discussed above are satisfied by the following transformation

$$\begin{aligned}\frac{\pi p}{2}\frac{dz}{dw} = &-i\left[\frac{I_1(1-A_1^2)}{(w-A_1)(A_1w-1)}+\frac{I_2(1-A_2^2)}{(w-A_2)(A_2w-1)}\right]\left[\log\left(\frac{w-e^{i\alpha}}{w-e^{-i\alpha}}\right)^2+i(3\pi-2\alpha)\right]\\ &+\frac{I_1(1+A_1)^2[-\pi+4\tan^{-1}\{(1+A_1)/(1-A_1)\tan\frac{1}{2}\alpha\}](w-1)}{(w-A_1)(A_1w-1)(w+1)}\\ &+\frac{I_2(1+A_2)^2[-\pi+4\tan^{-1}\{(1+A_2)/(1-A_2)\tan\frac{1}{2}\alpha\}](w-1)}{(w-A_2)(A_2w-1)(w+1)}.\end{aligned} \tag{14}$$

In the general case when A_1 and A_2 are complex and $\alpha_1 \neq -\alpha_2$ the transformation becomes

$$\begin{aligned}\frac{\pi p}{2}\frac{dz}{dw} = &-i\left[\frac{I_1(1-r_1^2)}{(w-A_1)(\bar{A}_1w-1)}+\frac{I_2(1-r_2^2)}{(w-A_2)(\bar{A}_2w-1)}\right]\\ &\times\left[\log\left(\frac{w-e^{i\alpha_1}}{w-e^{i\alpha_2}}\right)^2+i(3\pi-\alpha_1+\alpha_2)\right]-\frac{I_1(1-r_1^2)R_1(we^{i\delta_1}-e^{-i\delta_1})}{(w-A_1)(\bar{A}_1w-1)(w+1)}\\ &-\frac{I_2(1-r_2^2)R_2(we^{i\delta_2}-e^{-i\delta_2})}{(w-A_2)(\bar{A}_2w-1)(w+1)},\end{aligned} \tag{15}$$

where R_1, R_2, δ_1 and δ_2 are real and chosen so that (15) is regular at A_1 and A_2, that is

$$\frac{R_1(A_1 e^{i\delta_1} - e^{-i\delta_1})}{A_1+1} = 3\pi - \alpha_1 + \alpha_2 - i\log\left(\frac{A_1 - e^{i\alpha_1}}{A_1 - e^{i\alpha_2}}\right)^2, \tag{16}$$

and

$$\frac{R_2(A_2 e^{i\delta_2} - e^{-i\delta_2})}{A_2+1} = 3\pi - \alpha_1 + \alpha_2 - i\log\left(\frac{A_2 - e^{i\alpha_1}}{A_2 - e^{i\alpha_2}}\right)^2. \tag{17}$$

Two cuts are required to make the logarithmic expressions in the above equations single-valued. Thus cuts, at $e^{i\alpha_1}$ [for $\log(w - e^{i\alpha_1})$] and at $e^{i\alpha_2}$ [for $\log(w - e^{i\alpha_2})$] are made along the tangents to the unit circle at the corresponding points in the direction of θ increasing and of θ decreasing, respectively. (It is assumed that θ increases continuously from $-\pi$ to π.) Thus

$$2\operatorname{Arg}\left(\frac{A_1 - e^{i\alpha_1}}{A_1 - e^{i\alpha_2}}\right) = -4\pi + \alpha_1 - \alpha_2 + 2\tan^{-1}\left(\frac{1+r_1}{1-r_1}\tan\tfrac{1}{2}(\alpha_1 - \theta_1)\right)$$
$$- 2\tan^{-1}\left(\frac{1+r_1}{1-r_1}\tan\tfrac{1}{2}(\alpha_2 - \theta_1)\right). \tag{18}$$

The case of a plasma streaming past a two-dimensional dipole (Hurley 1961*b*) is obtained as the usual limit from (14) or (15).

We have to be careful as to what meaning we attach to the expression

$$\log\{(w - e^{i\alpha_1})/(w - e^{i\alpha_2})\}^2, \tag{19}$$

when (10) has no zeros (for examples when I_1 and I_2 are of the same sign). $e^{i\alpha_1}$ and $e^{i\alpha_2}$ are the points on the unit circle in the w-plane that correspond to the points at the boundary in the z-plane at which the magnetic field changes direction. These points may disappear either by moving first to infinity, in which case $\alpha_1 = -\alpha_2 = \pi$ and (19) is zero or by coinciding (when the distance between the two points in the z-plane vanishes). In this case $\alpha_1 = \alpha_2$ and the cuts made in the w-plane require that (19) be set equal to $-4\pi i$.

3. Results and discussion

Since in (10) dy is always of the same sign as $d\theta$ (except at $\theta = \alpha_1$ and $\theta = \alpha_2$, where $dy = 0$), it is not possible to arrange the position of the line currents with respect to the incident stream so as to get plasma trapped inside the field cavity as in the case of two line currents immersed in a stationary plasma at uniform pressure (Sozou 1966).

If we integrate (10) in the case when its right-hand side never vanishes, we see that at infinity the breadth of the cavity depends only on the intensity and not the position of the two line currents. From this it is easily deduced that at infinity the breadth of a cavity of any number of line currents immersed in a streaming plasma depends only on the algebraic sum of the currents enclosed, provided that just inside the boundary the magnetic field is always directed in the same sense. If the magnetic field at the boundary changes sign, the breadth of the cavity at infinity depends on the position of the two line currents inside the cavity. This breadth is a minimum if the two line currents are placed in such a position that the magnetic field at the boundary is always directed in the same sense. (This is always possible if the algebraic sum of the two line currents is not zero.)

If we integrate (15) and arrange that the origins in the two domains correspond we get four equations by equating real and imaginary parts for the

Curve	I_2/I_1	r_1	θ_1	X_1	Y_1	α_1	α_2
A	-1	0·99	$\frac{1}{2}\pi$	3·920	1·549	0	$-\pi$
B	-1	0·90	$\frac{1}{2}\pi$	1·625	1·366	0	$-\pi$
C	-1	0·75	$\frac{1}{2}\pi$	0·757	1·053	0	$-\pi$
D	-1	0·50	$\frac{1}{2}\pi$	0·207	0·547	0	$-\pi$
E	1	0·98	$\frac{1}{2}\pi$	3·245	1·549	—	—
F	1	0·90	$\frac{1}{2}\pi$	1·659	1·464	—	—
G	1	0·75	$\frac{1}{2}\pi$	0·827	1·286	—	—
H	1	0·50	$\frac{1}{2}\pi$	0·287	0·926	—	—
K	1	0·25	$\frac{1}{2}\pi$	0·064	0·489	—	—

TABLE 1

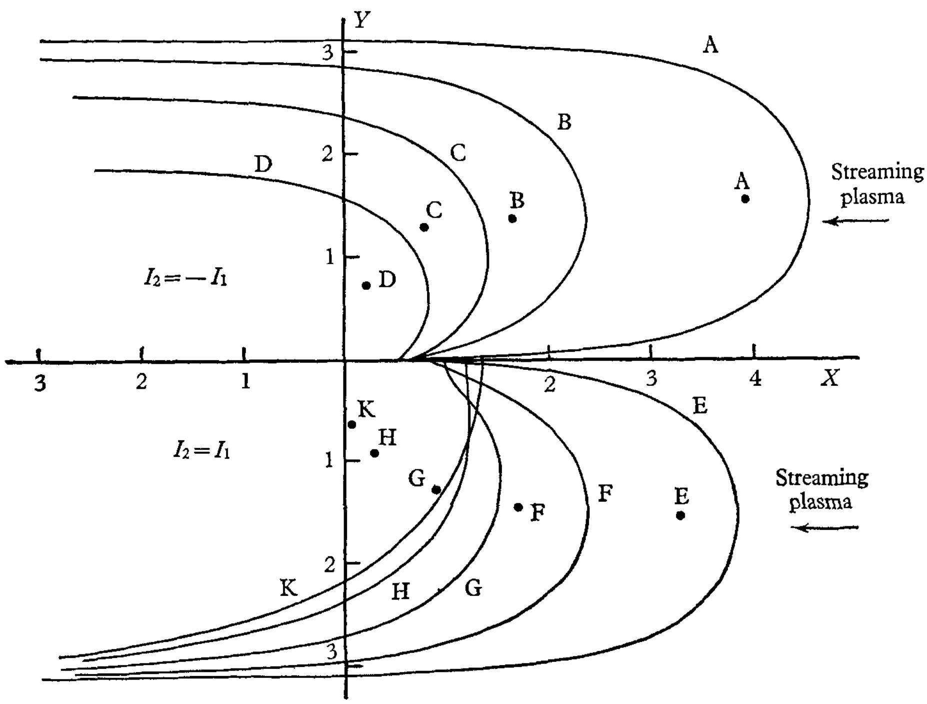

FIGURE 1. Half the cross-section of the boundary of the magnetic field of two line currents of the same intensity which are placed symmetrically in a streaming plasma. Letters indicate the position of one of the line currents and the corresponding boundary.

$$X = px/4\pi I_1, \quad Y = py/4\pi I_1.$$

positions of the two line currents. We get another four equations by equating real and imaginary parts in (16) and (17). By equating to zero the right-hand side of (10) we get two equations connecting α_1 and α_2 to r_1, r_2, θ_1 and θ_2. Thus we have ten equations in ten unknowns r_1, r_2, θ_1, θ_2, α_1, α_2, δ_1, δ_2, R_1 and R_2 whose determination, and substitution in the integrated form of (15) will give us the boundary in the physical plane. By choosing one of the two line currents to be at the origin the number of equations and unknowns is reduced by at least two. The simplest case arises when the two line currents are of the same sign, and the line joining them is parallel to the direction of the incident stream. In this case the required transformation is (14) with $\alpha_1 = \alpha_2 = 0$. If we further arrange so that

Curve	I_2/I_1	r_2	X_2	α_1	α_2
A	$-0{\cdot}25$	$0{\cdot}99$	$1{\cdot}54$	$0{\cdot}070$	$-0{\cdot}070$
B	$-0{\cdot}25$	$0{\cdot}90$	$0{\cdot}674$	$0{\cdot}204$	$-0{\cdot}204$
C	$-0{\cdot}25$	$0{\cdot}75$	$0{\cdot}346$	$0{\cdot}251$	$-0{\cdot}251$
D	$-0{\cdot}25$	$0{\cdot}50$	$0{\cdot}232$	—	—
E	$-0{\cdot}25$	$0{\cdot}25$	$0{\cdot}15$	—	—

TABLE 2

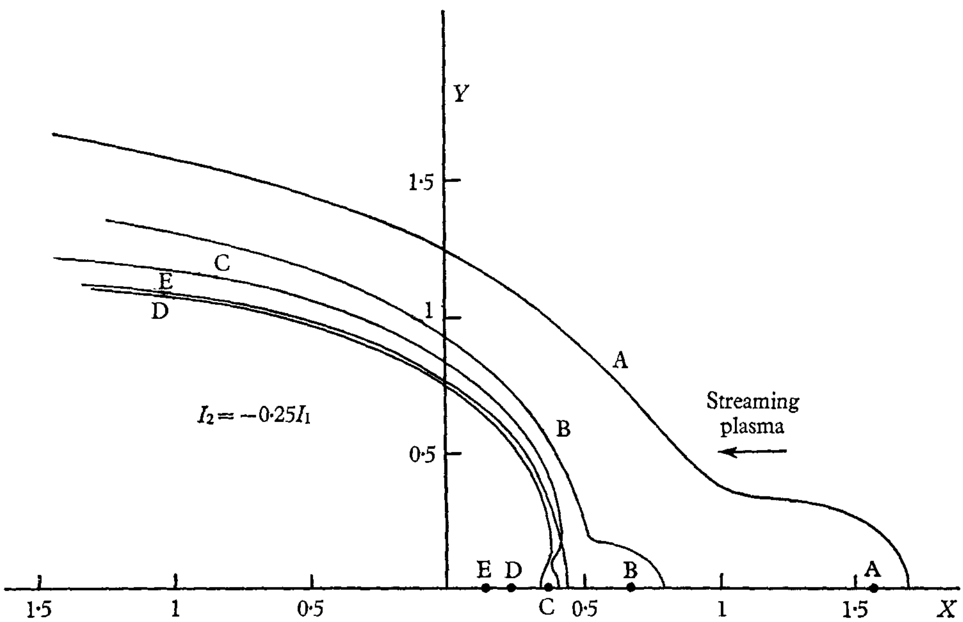

FIGURE 2. Half the cross-section of the boundary of the magnetic field of two line currents in line with the direction of the incident stream. I_1 is at the origin. Letters indicate the position of I_2 ($= -0{\cdot}25I_1$) and the corresponding boundary. $X = px/4\pi I_1$, $Y = py/4\pi I_1$.

one of the line currents, say I_1, is at the origin, we shall have to determine only one real unknown, A_2, the corresponding position of I_2 in the w-plane, in order that we may be able to calculate the boundary in the z-plane. A_2 is, of course, determined from the integrated form of (14). In the general case, however, the equations are very complicated. Thus it is much simpler if we assume values for r_1, r_2, θ_1 and θ_2 and obtain α_1 and α_2 from (10), and R_1, R_2, δ_1, and δ_2 from (16) and (17). The integrated form of (15) will then give us the boundary and the positions of the two line currents.

The method described above was programmed in Algol and the program was run on the Atlas computer of London University. Tables 1 and 2 show the sets of data for the corresponding boundaries of figures 1 and 2 respectively.

From figure 1 it is seen that as r_1, $r_2 \rightarrow 1$ the cavity tends to split into two cavities, one for each line current.

If the magnetic field changes direction at the boundary, in general, it does so at two pseudo-singularities in that there, only $dy = 0$ and $dz/dw \neq 0$. By a suitable choice of the data we may be able to get $dz/dw = 0$ at these points, that is, these points become singularities (cusps) of the boundary. These cusps are parallel to the direction of the incident stream and may allow plasma to enter the cavity. Note that in figure 1, at $y = 0$, the boundary is at right angles to the X-axis when $I_1 = I_2$ and touches it, that is, we have a cusp when $I_1 = -I_2$. When the boundary is symmetrical about the line joining the line currents, (14) shows that we get only one cusp when $\alpha = 0$. In this case the boundary magnetic field just becomes directed in the same sense, and there is a line of force through the singular point forming a closed loop round the nearest line current.

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